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Natural Frequency Sensitivity Analysis with Respect to Lumped Mass Location

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The calculation of natural frequencies for beams and plates with attached lumped masses is considered when the lumped masses are moved from a reference location. As such, the objective is to determine the natural frequency sensitivity to a change in the lumped mass location. The problem is solved using the Rayleigh quotient of the system in conjunction with Rayleigh's principle. The developed approach is straightforward to implement and yet has the capability of providing accurate predictions to changes in natural frequency even with substantial movement of a lumped mass from its original position. The formulation is developed for both distinct and repeated eigenvalues.

Nomenclature

A = cross-sectional area
$[A]^*$ = transverse shear stiffness matrix
a = x coordinate of the lumped mass
B = width of the plate
$[B_{,()}]$ = derivative matrix associated with repeated eigenvalue
b = y coordinate of the lumped mass
$[D]$ = bending stiffness matrix
E = Young's modulus
I = area moment of inertia about the neutral axis
L = length of the plate or beam
M = attached mass
r = number of eigenfunctions associated with a repeated eigenvalue
T = kinetic energy
$\mathcal{T}$ = coefficient of $(e^{i\Omega t})^2/2$ in T
t = thickness of plate
U = potential energy
$\mathcal{U}$ = coefficient of $(e^{i\Omega t})^2/2$ in U
W = transverse displacement trial function
$\hat{W}_0$ = matrix of eigenvectors associated with a repeated eigenvalue
w = transverse displacement
α = vector of constant coefficients
γ = vector of shear strains
δ = Dirac delta
κ = vector of bending strains
$\hat{\lambda}^N$ = new eigenvalue
$\hat{\lambda}^{Ne}$ = estimated eigenvalue
$\hat{\lambda}^O$ = original eigenvalue
ν = Poisson's ratio
ρ = material density
Ψ_x = section rotation about the y-axis trial function
Ψ_y = section rotation about the x-axis trial function
ψ_x = section rotation about the y axis
ψ_y = section rotation about the x axis
Ω = natural frequency

Subscripts

i = ith component of ( )
ij = i, j component of ( )
$,k$ = $\mathrm{d}()/\mathrm{d}k$ or $\partial()/\partial k$

Superscripts

$'$ = $\mathrm{d}()/\mathrm{d}x$
$''$ = $\mathrm{d}^2()/\mathrm{d}x^2$ or $\partial^2()/\partial x^2$

Introduction

THE subject of this work is the influence of lumped mass redistribution on the natural frequencies of structural systems. The importance of this topic is based on the observation that natural frequencies are some of the most significant parameters for control engineers and designers of dynamic structural systems and may be the dominant factor in the optimal design of these systems. Typical examples involving lumped mass redistribution are to be found in aircraft, robotic, large space structure, and printed circuit board design. Design and/or control optimality, almost by definition, requires design variable or configuration changes that in turn influences the natural frequencies of the system. Also, natural frequency evaluation involves the solution of an eigenvalue problem that for large systems can be computationally expensive; hence, it is desirable to have the capability of predicting the effects of variable changes on an eigenvalue without the requirement of solving an eigenvalue problem for every potential design variable change. This point of view leads to the desire for accurate, easily implemented design sensitivity analysis methodologies.

The design variables encountered in the present class of dynamic structural analyses can effectively be grouped into six categories. These are cross-sectional area,[1] material properties, lumped mass position,[2] support position,[2–4] boundary shape,[5] and layout variables.[6] There have been many eigenvalue sensitivity studies reported for all these categories except for the one dealing with the position of lumped masses.

With respect to the existing literature, a number of previous studies are relevant to the current work. Wang[2] has examined the eigenvalue sensitivity of an Euler–Bernoulli beam with respect to the position of a lumped mass using the classical modal summation method. In addition, studies completed for plates with concentrated masses include the work by Whaley,[7] Pombo et al.,[8] and Nicholson and Bergman,[9] and the references therein. It should be mentioned that the work by Whaley[7] concentrates on the effect of the introduction of a lumped mass into a system that originally has no lumped mass.

The problem considered in this paper assumes that lumped masses are present in a structure and furthermore the eigenpairs (i.e., eigenfunctions and eigenvalues) for this reference structure have been determined. The question posed is to determine the change in the eigenvalues with respect to a perturbation in the position of the lumped masses. The objectives of this paper are twofold. First, the work by Wang[2] is reformulated based on the Rayleigh quotient[3]; the current approach is shown to yield identical results by a simpler technique. Second, the current method is extended to plate vibration problems. The formulation is evaluated for two-plate problems by

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comparing the results of the sensitivity calculation to results obtained from a complete reanalysis for the perturbed problem. All results are obtained using the finite element method.

Mathematical Derivation

Two problems will be considered, beams and plates with attached lumped masses. The problems will be considered in sequence.

Beam with a Lumped Mass Attachment

A cantilever beam with a lumped mass attached at a point along its span is illustrated in Fig. 1. The reference frame XYZ is as shown and it is assumed that the mass is attached at the point a along the x axis of the reference frame. The beam is modeled using Euler–Bernoulli beam theory.

The strain-energy of the beam is the only contributor to the potential energy of the system U whereas the kinetic energy T is comprised of components from both the beam and the lumped mass. These quantities are given, respectively, as

$$U = \frac{1}{2}\int_0^L EI w''^2 \,\mathrm{d}x$$
$$T = \frac{1}{2}\int_0^L \rho A \dot{w}^2 \,\mathrm{d}x + \int_0^L M\dot{w}^2\delta(x-a)\,\mathrm{d}x \tag{1}$$

The solution $w(x,t)$ of the free-vibration problem is written as

$$w(x,t) = W(x)e^{i\Omega t} \tag{2}$$

Substitution of this expression into the kinetic and potential energies and forming the Rayleigh quotient $R[W(x)]$ yields

$$R[W(x)] = \frac{\mathcal{U}[W(x)]}{\mathcal{T}[W(x)]} \tag{3}$$

where $\mathcal{U}[W(x)]$ and $\mathcal{T}[W(x)]$ are the coefficients of $(e^{i\Omega t})^2/2$ corresponding to the potential and kinetic energies respectively and are given by

$$\mathcal{U}[W(x)] = \int_0^L EIW''(x)^2\,\mathrm{d}x$$
$$\mathcal{T}[W(x)] = \int_0^L \rho A W(x)^2\,\mathrm{d}x + \int_0^L MW(x)^2\delta(x-a)\,\mathrm{d}x \tag{4}$$

Rayleigh's principle provides the result that the stationary value of the Rayleigh quotient yields the system eigenvalue λ (i.e., square of the natural frequency) and this occurs when the argument function $W(x)$ is the corresponding eigenfunction.[10] Thus the eigenvalue $\hat{\lambda}$ is found from

$$\hat{\lambda} = \mathrm{St}\,R[W(x)] = \mathrm{St}\frac{\mathcal{U}[W(x)]}{\mathcal{T}[W(x)]} \tag{5}$$

subject to the conditions that the admissible functions $W(x)$ are sufficiently differentiable with respect to x and satisfy the appropriate kinematic boundary conditions. The abbreviation St implies stationary condition.

If a particular eigenpair is denoted by $\hat{\lambda}$, $\hat{W}(x)$, Eq. (5) can be rewritten as

$$\hat{\lambda} = R[\hat{W}(x)] = \frac{U[\hat{W}(x)]}{T[\hat{W}(x)]} = \frac{\int_0^L EI[\hat{W}''(x)]^2\,\mathrm{d}x}{\int_0^L[\rho A + M\delta(x-a)]\hat{W}^2(x)\,\mathrm{d}x} \tag{6}$$

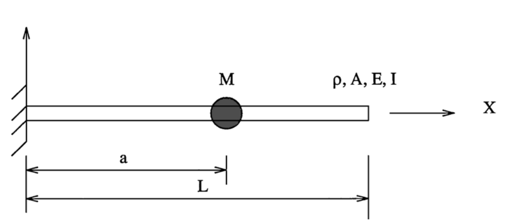

Fig. 1 Schematic of a beam with an attached lumped mass.

Note in the preceding equation that $\hat{\lambda}$ is a function of the position of the lumped mass a. In addition, the differentiability requirement and the boundary conditions imposed on the eigenfunction are independent of the lumped mass location. Hence, the eigenvalue sensitivity with respect to the lumped mass location is obtained by differentiating Eq. (6) with respect to a. This yields

$$\frac{\mathrm{d}\hat{\lambda}}{\mathrm{d}a} = \left[-\hat{\lambda}\int_0^L M\frac{\mathrm{d}\delta(x-a)}{\mathrm{d}a}\hat{W}^2(x)\,\mathrm{d}x\right]\Big/ T[\hat{W}(x)] \tag{7}$$

In obtaining this result, it is observed that whereas $\hat{W}(x)$ is implicitly dependent on the lumped mass location a, the variation of $\hat{W}(x)$ with respect to a does not contribute a first-order effect to the eigenvalue sensitivity because $R[\hat{W}(x)]$ is stationary with respect to $\hat{W}(x)$.

From the calculus of δ functions,[11]

$$\int_0^L f(x)\frac{\mathrm{d}\delta(x-a)}{\mathrm{d}a}\,\mathrm{d}x = \int_0^L \frac{\mathrm{d}f(x)}{\mathrm{d}x}\delta(x-a)\,\mathrm{d}x = f'(a) \tag{8}$$

allows Eq. (7) to be written as

$$\frac{\mathrm{d}\hat{\lambda}}{\mathrm{d}a} = -\frac{2\hat{\lambda}M\hat{W}(a)\hat{W}'(a)}{T[\hat{W}(x)]} \tag{9}$$

This result is equivalent to Eq. (14) in the paper by Wang,[2] although in the present case the derivation is more straightforward.

With respect to the result of Eq. (9), note that the product $\hat{\lambda}M\hat{W}(a)$ can be interpreted as an equivalent force corresponding to the inertia of the lumped mass, i.e., D'Alembert's principle. Hence the eigenvalue sensitivity is proportional to this inertial force times the slope of the eigenfunction at the mass attachment point.

The derivation presented thus far is valid for distinct eigenvalues. The solution for repeated eigenvalues is different in that a subeigenproblem[12] must be solved. To illustrate the technique, assume an r-fold repeated eigenvalue with an associated set of r linearly independent eigenfunctions $\{\hat{W}_1, \hat{W}_2, \ldots, \hat{W}_r\}$. Because of the linear independence of these eigenfunctions, any linear combination of this set is also an eigenfunction corresponding to the repeated eigenvalue. Such an eigenfunction $\hat{W}_0$ may be expressed as

$$\hat{W}_0 = [\hat{W}_1 \quad \hat{W}_2 \quad \cdots \quad \hat{W}_r]\boldsymbol{\alpha} \tag{10}$$

where $\boldsymbol{\alpha}$ is a vector that is to be determined.

Substituting Eq. (10) into Eq. (5) and seeking a stationary solution with respect to $\boldsymbol{\alpha}$ yields

$$\hat{\lambda} = \mathrm{St}\,R[\hat{W}_0] = \mathrm{St}\left[\frac{\boldsymbol{\alpha}^T[A]\boldsymbol{\alpha}}{\boldsymbol{\alpha}^T[B]\boldsymbol{\alpha}}\right] \tag{11}$$

where $[A]$ and $[B]$ are $r\times r$ matrices with elements given by

$$A_{ij} = \int_0^L EI\hat{W}_i''(x)\hat{W}_j''(x)\,\mathrm{d}x \tag{12}$$

$$B_{ij} = \int_0^L [\rho A + M\delta(x-a)]\hat{W}_i(x)\hat{W}_j(x)\,\mathrm{d}x \tag{13}$$

Differentiation of Eq. (11) with respect to a yields

$$\frac{\mathrm{d}\hat{\lambda}}{\mathrm{d}a} = -\hat{\lambda}\,\mathrm{St}\left[\frac{\boldsymbol{\alpha}^T[B']\boldsymbol{\alpha}}{\boldsymbol{\alpha}^T[B]\boldsymbol{\alpha}}\right] \tag{14}$$

where $[B']$ is an $r\times r$ matrix with entries given by

$$B'_{ij} = M[\hat{W}_i'(a)\hat{W}_j(a) + \hat{W}_i(a)\hat{W}_j'(a)] \tag{15}$$

The stationary problem of Eq. (14) is thus transformed into the eigenproblem

$$\hat{\lambda}[B']\boldsymbol{\alpha} = \frac{\mathrm{d}\hat{\lambda}}{\mathrm{d}a}[B]\boldsymbol{\alpha} \tag{16}$$

from which the eigenvalue sensitivity can be obtained. It is to be noted that the eigenvalue sensitivity is now a directional derivative with r components corresponding to the solution of this eigenproblem. $[B]$ is the identity matrix because of the orthonormalization

Table 1 Eigenvalue sensitivity with respect to lumped mass location on a simply supported square plate $\Delta b = 0.125$ m

Mode number	Eigenvalue $\times 10^3$, $\hat{\lambda}^O$	Eigenvalue $\times 10^3$, $\hat{\lambda}^N$	Eigenfunction $\times 10^{-2}$, $\hat{W}(a,b)$	Eigenfunction slope $\times 10^{-2}$, FFD	Eigenvalue rate $\times 10^3$, $(\hat{\lambda}^N - \hat{\lambda}^O)/\Delta b$	Eigenvalue rate $\times 10^3$, $-2\hat{\lambda}^O \hat{W}(a,b) \times \hat{W}_{,y}(a,b)M$, FFD	Estimated eigenvalue $\times 10^3$, $\hat{\lambda}^{Ne} \approx \hat{\lambda}^O + (\mathrm{d}\hat{\lambda}/\mathrm{d}b)\Delta b$
1	4.627	4.180	2.464	1.446	−3.576	−3.297	4.215
2	31.003	34.435	1.450	−3.879	27.456	34.872	35.362
3	63.881	63.881	0	0	0	0	63.881

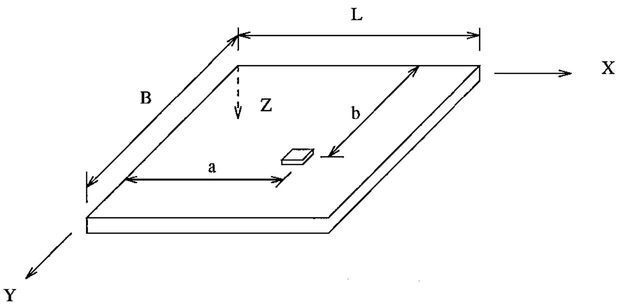

Fig. 2 Schematic of a plate with an attached lumped mass.

of the eigenvectors with respect to the inertia matrix $[M]$; however, $[B']$ is in general fully populated.

Plate with a Lumped Mass Attachment

In this section, the work is extended to a Reissner–Mindlin plate with an attached lumped mass (Fig. 2).

The potential energy U of the Reissner–Mindlin plate is

$$U[w, \psi_x, \psi_y] = \frac{1}{2}\int_A [\boldsymbol{\kappa}^T \quad \boldsymbol{\gamma}^T] \begin{bmatrix} [D] & 0 \\ 0 & [A]^* \end{bmatrix} \begin{Bmatrix} \boldsymbol{\kappa} \\ \boldsymbol{\gamma} \end{Bmatrix} \mathrm{d}A \tag{17}$$

where $[D]$ is the flexural stiffness matrix and $[A]^*$ is the transverse shear stiffness matrix. Also, $\boldsymbol{\kappa}$ and $\boldsymbol{\gamma}$ are given by

$$\boldsymbol{\kappa} = \begin{Bmatrix} \dfrac{\partial \psi_x}{\partial x} \\ \dfrac{\partial \psi_y}{\partial y} \\ \dfrac{\partial \psi_x}{\partial y} + \dfrac{\partial \psi_y}{\partial x} \end{Bmatrix}, \qquad \boldsymbol{\gamma} = \begin{Bmatrix} \dfrac{\partial w}{\partial y} - \psi_y \\ \dfrac{\partial w}{\partial x} - \psi_x \end{Bmatrix}$$

The kinetic energy T is

$$T = \frac{1}{2}\int_A \rho A \dot{w}^2 \,\mathrm{d}A + \frac{1}{2}\int_A \rho I\left(\dot{\psi}_x^2 + \dot{\psi}_y^2\right) \mathrm{d}A + \frac{1}{2}\int_A M\dot{w}^2 \delta(x-a)\delta(y-b)\,\mathrm{d}A \tag{18}$$

The solution of the free-vibration problem is expressed as

$$w(x,y,t) = W(x,y)e^{i\Omega t}, \qquad \psi_x(x,y,t) = \Psi_x(x,y)e^{i\Omega t}$$

$$\psi_y(x,y,t) = \Psi_y(x,y)e^{i\Omega t}$$

Following the same procedure as before, the eigenvalues corresponding to λ are the stationary values of the Rayleigh quotient and the arguments $W(x,y)$, $\Psi_x(x,y)$ and $\Psi_y(x,y)$ corresponding to these minima are the eigenfunctions.[10] That is,

$$\lambda = \mathrm{St}\frac{\mathcal{U}[W(x,y), \Psi_x(x,y), \Psi_y(x,y)]}{\mathcal{T}[W, \Psi_x(x,y), \Psi_y(x,y)]} \tag{19}$$

where the field variables or eigenfunctions $W(x,y)$, $\Psi_x(x,y)$, and $\Psi_y(x,y)$ are continuous and differentiable.

The stationarity condition of the Rayleigh quotient corresponding to the eigenvalue $\hat{\lambda}$ and eigenfunction $\hat{W}(x,y)$, $\hat{\Psi}_x(x,y)$, $\hat{\Psi}_y(x,y)$ is therefore

$$\hat{\lambda} = \frac{\mathcal{U}[\hat{W}(x,y), \hat{\Psi}_x(x,y), \hat{\Psi}_y(x,y)]}{\mathcal{T}[\hat{W}(x,y), \hat{\Psi}_x(x,y), \hat{\Psi}_y(x,y)]} \tag{20}$$

The eigenvalue sensitivity is obtained by differentiating the preceding equation with respect to a and b, which yields the results

$$\frac{\mathrm{d}\hat{\lambda}}{\mathrm{d}a} = -\frac{2\hat{\lambda} M \hat{W}(a,b)\hat{W}_{,x}(a,b)}{\mathcal{T}[\hat{W}(x,y), \hat{\Psi}_x(x,y), \hat{\Psi}_y(x,y)]}$$
$$\frac{\mathrm{d}\hat{\lambda}}{\mathrm{d}b} = -\frac{2\hat{\lambda} M \hat{W}(a,b)\hat{W}_{,y}(a,b)}{\mathcal{T}[\hat{W}(x,y), \hat{\Psi}_x(x,y), \hat{\Psi}_y(x,y)]} \tag{21}$$

where $\hat{W}_{,x}(a,b)$, $\hat{W}_{,y}(a,b)$ are the partial derivatives of $\hat{W}(x,y)$ with respect to x and y, respectively, evaluated at (a,b).

Normalizing the eigenfunctions such that $\mathcal{T} = 1$ allows Eq. (21) to be rewritten as

$$\frac{\mathrm{d}\hat{\lambda}}{\mathrm{d}a} = -2\hat{\lambda} M \hat{W}(a,b)\hat{W}_{,x}(a,b)$$
$$\frac{\mathrm{d}\hat{\lambda}}{\mathrm{d}b} = -2\hat{\lambda} M \hat{W}(a,b)\hat{W}_{,y}(a,b) \tag{22}$$

As was the case for the beam problem, it is noted that the product $\hat{\lambda} M \hat{W}(a,b)$ in Eq. (22) can be interpreted as the inertia of the lumped mass, and therefore the eigenvalue sensitivity is proportional to this inertial force times the slope of the eigenfunction at the mass attached point.

The sensitivity formulation for the case of repeated eigenvalues follows the outline presented for the beam and leads to a pair of similar subeigenproblems:

$$\hat{\lambda}[B_{,a}]\boldsymbol{\alpha} = \frac{\mathrm{d}\hat{\lambda}}{\mathrm{d}a}[B]\boldsymbol{\alpha}, \qquad \hat{\lambda}[B_{,b}]\boldsymbol{\alpha} = \frac{\mathrm{d}\hat{\lambda}}{\mathrm{d}b}[B]\boldsymbol{\alpha} \tag{23}$$

where $[B]$, $[B_{,a}]$ and $[B_{,b}]$ are $r \times r$ matrices with elements given by

$$B_{ij} = \int_A [\rho A \hat{W}_i \hat{W}_j + \rho(\hat{\Psi}_{xi}\hat{\Psi}_{xj} + \hat{\Psi}_{yi}\hat{\Psi}_{yj}) + M\hat{W}_i \hat{W}_j \delta(x-a)\delta(y-b)]\,\mathrm{d}A \tag{24}$$

$$B_{ij,a} = M[\hat{W}_{i,x}(a,b)\hat{W}_j(a,b) + \hat{W}_i(a,b)\hat{W}_{j,x}(a,b)] \tag{25}$$

$$B_{ij,b} = M[\hat{W}_{i,y}(a,b)\hat{W}_j(a,b) + \hat{W}_i(a,b)\hat{W}_{j,y}(a,b)] \tag{26}$$

Numerical Examples

Two plate-lumped mass configurations will be examined; each is based on a square plate with length $L = 3$ m, thickness $t = 0.03$ m, Young's modulus $E = 2.07 \times 10^{11}$ N m^{-2}, Poisson's ratio $\nu = 0.3$, and density $\rho = 8.0 \times 10^3$ kg m^{-3}.

In the first, the eigenvalue sensitivity is evaluated for two cases of distinct eigenvalues corresponding to simply supported and fully

Table 2 Eigenvalue sensitivity with respect to lumped mass location on a clamped square plate; $\Delta b = 0.125$ m

Mode number	Eigenvalue $\times 10^3$		Eigenfunction $\times 10^{-2}$,	Eigenfunction slope $\times 10^{-2}$,	Eigenvalue rate $\times 10^3$		Estimated eigenvalue $\times 10^3$,
	$\hat{\lambda}^O$	$\hat{\lambda}^N$	$\hat{W}(a,b)$	FFD	$(\hat{\lambda}^N - \hat{\lambda}^O)/\Delta b$	$-2\hat{\lambda}^O \hat{W}(a,b)\times \hat{W}_{,y}(a,b)M$, FFD	$\hat{\lambda}^{Ne} \approx \hat{\lambda}^O + (\mathrm{d}\hat{\lambda}/\mathrm{d}b)\Delta b$
1	13.126	11.150	2.692	1.877	−15.808	−13.270	11.467
2	63.360	70.945	1.215	−5.302	60.680	81.611	73.561
3	141.360	141.360	0	0	0	0	141.360

Table 3 Sensitivity of distinct and repeated eigenvalues with respect to lumped mass location on a simply supported square plate; $\Delta b = 0.125$ m

Mode number	Eigenvalue $\times 10^3$		Eigenfunction $\times 10^{-2}$,	Eigenfunction slope $\times 10^{-2}$,	Eigenvalue rate $\times 10^3$		Estimated eigenvalue $\times 10^3$,
	$\hat{\lambda}^O$	$\hat{\lambda}^N$	$\hat{W}(a,b)$	FFD	$(\hat{\lambda}^N - \hat{\lambda}^O)/\Delta b$	$-2\hat{\lambda}^O \hat{W}(a,b)\times \hat{W}_{,y}(a,b)M$, FFD	$\hat{\lambda}^{Ne} \approx \hat{\lambda}^O + (\mathrm{d}\hat{\lambda}/\mathrm{d}b)\Delta b$
1	5.1433	4.995	1.600	1.316	−1.187	−1.083	5.008
2	20.047	19.481	2.209	0.641	−4.528	−2.848	19.694
3	20.047	20.264	−1.517	0.881	1.736	3.008	20.640

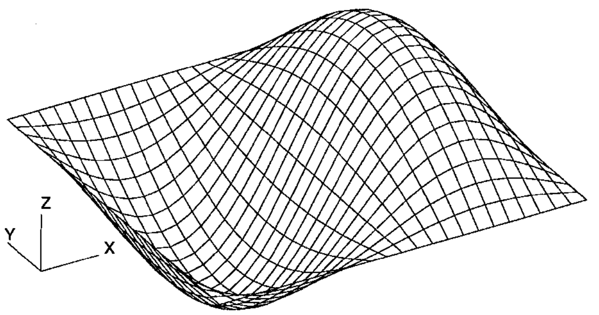

Fig. 3 Mode shape of the third eigenvalue for the simply supported square plate.

clamped boundary conditions. A lumped mass $M = 1000$ kg is attached at the location ($a = 1.5$ m, $b = 0.75$ m). For the simply supported plate, the first three eigenvalues $\hat{\lambda}^O$ are tabulated in the second column of Table 1. The results are obtained from a finite element analysis using 64 bicubic Lagrange elements. The lumped mass is then moved to the position ($a = 1.5$ m, $b = 0.875$ m); thus $\Delta b = 0.125$ m, and the first three new eigenvalues $\hat{\lambda}^N$ are determined and presented in the third column of Table 1. The fourth column provides the value of the appropriate eigenfunction at the lumped mass location. The fifth column indicates that the slope $\hat{W}_{,x}$ or $\hat{W}_{,y}$ is computed using forward finite differences (FFD). The finite difference calculation is required because the eigenfunction slopes at the original position of the lumped mass cannot be determined directly from the finite element analysis (i.e., as a degree of freedom) because Reissner–Mindlin theory does not enforce solution slope continuity across element boundaries. The remaining entries in the table are self-explanatory. The results from the analysis of the fully clamped plate are given in Table 2.

The results presented in Tables 1 and 2 illustrate that the calculated eigenvalue sensitivity gives a good estimate of $(\hat{\lambda}^N - \hat{\lambda}^O)/\Delta b$. A plausible reason for the observed differences is the finite size of the displacement Δb used; a very small value Δb (Ref. 3) is expected to result in a very accurate estimate but such a movement is not practical or realistic and hence explains the choice of $\Delta b = 0.125$ m. It may be noted that the absence of change in the eigenvalue corresponding to the third mode for both the clamped and the simply supported scenarios is because the mass is located on a nodal line of the eigenmode. Figure 3 illustrates this solution characteristic for the third mode of the simply supported case.

The second configuration illustrates the use of the present formulation for the case of repeated eigenvalues. Here four 500-kg masses are symmetrically attached to the simply supported square plate at the locations (0.75 m, 0.75 m), (0.75 m, 2.25 m), (2.25 m, 0.75 m), and (2.25 m, 2.25 m). The problem is to determine the sensitivity of the first three eigenvalues when the mass at (0.75 m, 0.75 m) is moved along the y axis. For comparison purposes, calculations are completed when $\Delta b = 0.125$ m.

The first mode is distinct and the eigenvalue sensitivity is observed to be accurately estimated. The second and third modes have a repeated eigenvalue, and their sensitivities are as tabulated in Table 3. Although there is a large difference between these and those obtained from the finite element method [i.e., $(\hat{\lambda}^N - \hat{\lambda}^O)/\Delta b$], the error in the actual eigenvalue (i.e., $\hat{\lambda}^N$ and $\hat{\lambda}^{Ne}$) is less than 1%. The discrepancies, like those observed in the first scenario, are a result of the finite size of Δb.

Conclusion

A formulation is presented for the sensitivity calculation of both distinct and repeated eigenvalues of beams and plates with respect to the location of a lumped mass. The approach has been shown to be straightforward and effective, and it has been illustrated to provide very good estimates of eigenvalue sensitivities even when the displacement from the original location of the lumped mass is substantial.

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